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ON THE SECONDARY INSTABILITY OF THE MOST DANGEROUS GÖRTLER VORTEX.

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ABSTRACT

Recent studies have demonstrated the most unstable Görtler vortex mode is found in flows, both two and three-dimensional, with regions of (moderately) large body curvature and these modes reside within a thin layer situated at the base of the conventional boundary layer. Further work concerning the nonlinear development of the most dangerous mode demonstrates that the flow results in a self induced flow reversal. However, prior to the point at which flow reversal is encountered the total streamwise velocity profile is found to be highly inflectional in nature. Previous work then suggests that the nonlinear vortex state will become unstable to secondary, inviscid, Rayleigh wave instabilities prior to the point of flow reversal. Our concern is with the secondary instability of the nonlinear vortex states, which result from the streamwise evolution of the most unstable Görtler vortex mode, with the aim of determining whether such modes can induce a transition to a fully turbulent state before separation is encountered.

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1. Introduction

The most dangerous Görtler vortex mode has recently been identified by Denier et al (1991) (see also Timoshin (1990)). This mode is found to have a streamwise growth rate of $O(G^{3/5})$ and is localized in a thin, $O(G^{-1/5})$, viscous layer located at the solid boundary. The effect of crossflow on this mode has been demonstrated by Bassom & Hall (1991) to have a stabilizing effect. The nonlinear evolution of the most dangerous mode has been considered in a series of articles; for two dimensional flows by Denier & Hall (1993) and for three dimensional flows by Otto & Bassom (1993). In both cases the vortex induced mean flow results in a region of reversed flow at some finite distance downstream of the position at which the perturbation is first introduced. However, prior to such a flow reversal the total streamwise velocity fields are seen to be strongly inflectional in nature and thus we anticipate that such flows will be highly susceptible to secondary inviscid instabilities in the form of Rayleigh waves.

In the case of order one wavenumber/Görtler number regime Hall & Horseman (1991) found that the highly inflectional velocity profiles found in Hall (1988) are unstable to inviscid Rayleigh instabilities and demonstrated good agreement with the experimental results of Swearingen & Blackwelder (1987).

In the situation under consideration the equation governing the perturbation quantities is the three–dimensional Rayleigh pressure equation obtained by Hall & Horseman (1991). We obtain solutions of this boundary value problem in order to ascertain the temporal stability of a given vortex flow profile. Since the presence of the vortex implies that the underlying basic state has a periodic spanwise structure, we elect to retain the spanwise variation by only assuming that the periodicity of the secondary modes are the same as the that of the vortices. In fact we find that the most unstable modes are comprised of significant contributions from the mean, fundamental and second harmonic components. Similar observations have been made by Balachandar, Streett & Malik (1992), in their work concerning the secondary instability of rotating disk flows. Somewhat surprisingly this procedure does not generate any *odd* modes, those which are $\pi/2$ out–of–phase with the underlying vortex state, of the form found by Hall & Horseman (1991). However, by explicitly apply spanwise boundary conditions applicable to the odd modes additional unstable modes can readily be obtained but which have smaller growth rates than the those labeled secondary in Figure 1, (see Otto & Denier (1993) for further details).

The remainder of this article is structured as follows: in section 2 we shall summarize the derivation of the equations governing the nonlinear vortex state, and derive the inviscid Rayleigh pressure equation. In section 3 some brief comments will be made concerning the numerical methods used to solve the Rayleigh equation, and in section 4 we shall comment on our findings and discuss possible future topics of interest, and finally in section 5 we shall draw some conclusions.

2. Governing equations

In this article we consider a boundary layer flowing over a yawed cylinder, this flow has been previously considered in Hall (1985) and the reader is referred to that paper for details. The Reynolds number R_e and Görtler number G are defined by

$$R_e = \frac{UL}{\nu}, \qquad G = 2R_e^{\frac{1}{2}}\delta,$$

where U is a typical flow velocity in the streamwise direction, L is a characteristic streamwise lengthscale and ν is the kinematic viscosity of the fluid. The curvature of the cylinder is taken to be $\frac{1}{b}\chi_0\left(\frac{x}{L}\right)$ where the function χ_0 is supposed to be smooth and positive. With these definitions $\delta \equiv L/b$ where b is a typical radius of curvature of the cylinder. The Reynolds number is taken to be large whilst δ is sufficiently small so that as $\delta \to 0$ the parameter G is fixed and is of order one (when compared to the Reynolds number). We will subsequently consider the large Görtler number limit relevant to the most dangerous Görtler vortex mode.

The nonlinear evolution of the most dangerous Görtler mode occurs over an $(G^{-3/5})$ spatial lengthscale, (where now we are assuming $G \gg 1$), and is confined to an $O(G^{-1/5})$ viscous layer located at the solid boundary. With the effect of crossflow of order $O(R_e^{-\frac{1}{2}}G^{\frac{3}{5}})$ the modes evolve over an $O(G^{-\frac{2}{5}})$ temporal scale. The resulting equations governing the nonlinear evolution of the most dangerous Görtler mode are given in Otto & Bassom (1993) (also Denier & Hall (1993), Timoshin (1990)). For the sake of brevity the reader is referred to the aforementioned papers for full details.

Here we make a few brief remarks concerning the results of the previous calculations of the nonlinear evolution of the most unstable Görtler mode. The work of Denier & Hall (1993) demonstrates that the evolution of this mode results in a self induced flow reversal at some finite distance downstream of the position at which a disturbance is first introduced into the boundary layer. This result was independently confirmed by Otto & Bassom (1993) in their work on the effect of crossflow on the nonlinear evolution of the most dangerous Görtler mode. However, in both cases, it was demonstrated that the total flow field becomes highly inflectional in nature prior to the point at which flow reversal is found.

To consider the secondary instability of such highly inflectional profiles we consider perturbations to the total velocity field in the form of inviscid Rayleigh waves. The spatial and temporal scales of these modes are on the same order as the viscous wall sublayer in which the nonlinear vortex resides, namely $O(G^{-1/5}Re^{-1/2})$. We consider perturbations to the nonlinear vortex state of the form

$$\Delta\left(G^{\frac{1}{5}}\hat{U}, G^{\frac{1}{5}}\hat{V}, G^{\frac{1}{5}}\hat{W}, \hat{P}\right)e^{i\alpha G^{1/5}Re^{1/2}(x-ct)},$$

where α is the streamwise wave number and c is the complex wave speed; here $\hat{U}, \hat{V}, \hat{W}, \hat{P}$ are functions of the scaled variables y and z and Δ is the small perturbation parameter. The equation governing the perturbation quantities are most easily written by eliminating $\hat{U}, \hat{V}, \hat{W}$ to give the three-dimensional Rayleigh pressure equation

$$\frac{\partial^2 \hat{P}}{\partial y^2} + \frac{\partial^2 \hat{P}}{\partial z^2} - \alpha^2 \hat{P} - \frac{2}{\bar{u}-c}\left(\frac{\partial \bar{u}}{\partial y}\frac{\partial \hat{P}}{\partial y} + \frac{\partial \bar{u}}{\partial z}\frac{\partial \hat{P}}{\partial z}\right) = 0. \qquad (2.1)$$

We impose the usual inviscid boundary conditions, namely vanishing normal velocity at the solid boundary together with the requirement that the perturbation decays as we leave the viscous wall layer. In terms of the pressure perturbation these requirements become

$$\frac{\partial \hat{P}}{\partial y} = 0 \quad \text{at} \quad y = 0, \qquad \hat{P} \to 0 \quad \text{as} \quad y \to \infty. \qquad (2.2)$$

We also impose the condition that the wave has the same spanwise period as the underlying nonlinear vortex state, hence

$$\hat{P}(z) = \hat{P}(z + 2\pi), \qquad (2.3)$$

where k is the spanwise wavenumber of the vortex velocity field. The form of $\bar{u}$ is given by the total streamwise velocity from the vortex calculation, and is

$$\bar{u}(x, y, z) = y + \tilde{U}(x, y, z), \qquad (2.4)$$

with $\bar{u}(z) = \bar{u}(z + 2\pi)$; the term y in relation (2.4) is the basic shear in the viscous wall layer and U is the vortex induced flow. This system now represents an eigenvalue problem which can be solved for a given α to determine a complex phase speed c and thus determine the inviscid temporal instability of the flow at a given streamwise, x location.

3. Numerical techniques

In this section we describe the techniques used to solve the elliptic equation (2.1). We discretize the system using a five-point regular spanwise stencil and a stretched three point stencil in the normal coordinate. As the Rayleigh waves are presumed to have the same period (or integer divisors thereof) as the vortex, we alias the point at $z = 2\pi$ to have the same value as that at $z = 0$. The stretching in the normal coordinate ensures that the far field boundary condition can be imposed at a suitably large (but finite) value while still retaining resolution at the wall. The discretized version of equation (2.1) may be written as

$$\mathbf{A}_i \mathcal{P}_{i+1} + \mathbf{B}_i \mathcal{P}_i + \mathbf{C}_i \mathcal{P}_{i-1} = 0, \tag{3.1}$$

where $\mathcal{P}_i = \left(\hat{P}_{i1}, \hat{P}_{i2}, \dots \hat{P}_{iM}\right)^T$, where the subscript i denotes the value at y_i. In the system (3.1) the matrices $\mathbf{A}_i$ and $\mathbf{C}_i$ are diagonal, a fact that is exploited by the particular block solver written to solve (3.1). To impose the asymptotic condition we notice that $\hat{P}$ satisfies

$$\frac{\partial^2 \hat{P}}{\partial y^2} - \frac{2}{y-c}\frac{\partial \hat{P}}{\partial y} - \alpha^2 \hat{P} = 0,$$

as $y \to \infty$, where we have made use of the fact that the vortex is confined to the viscous wall layer. This equation has the decaying solution

$$\hat{P} = (y + A)\, e^{-\alpha y},$$

where A is a function of α and c but is not relevant here. We choose to impose this condition at $y = y_\infty$ by using the Robin condition,

$$\frac{1}{\hat{P}}\frac{\partial \hat{P}}{\partial y} = -\alpha + \frac{1}{y}.$$

This outer limit is chosen so that changing it does not affect the calculation. In this study we choose to allow both odd and even modes, and thus we solve over the whole period rather than half the range as in Hall & Horseman (1991). This allowed us to have mixed modes which are necessary in the three-dimensional basic states cases. The problem was normalized using the same method as in Hall & Horseman (1991) by imposing the constraint

$$\frac{\partial \hat{P}}{\partial y} = 1 \qquad \text{at} \qquad y = 0 \qquad \forall z. \tag{3.2}$$

We then iterate on the complex phase speed c to drive the complex number,

$$\frac{1}{\int \text{Real}\left(\hat{P}^2|_{y=0}\right) + i\text{Imag}\left(\hat{P}^2|_{y=0}\right) dZ},$$

to zero, where we use a two-dimensional real secant method. When the system is renormalized the condition (3.2) is in fact $\hat{P}_y = 0$, at $y = 0$.

In general 32 points per period were used in the spanwise coordinate whereas the normal grid was made up of 90 points, with an infinity of $y_\infty = 50$. In general the same number of points were used for these calculations as were employed in Otto & Bassom (1993) for the vortex calculations.

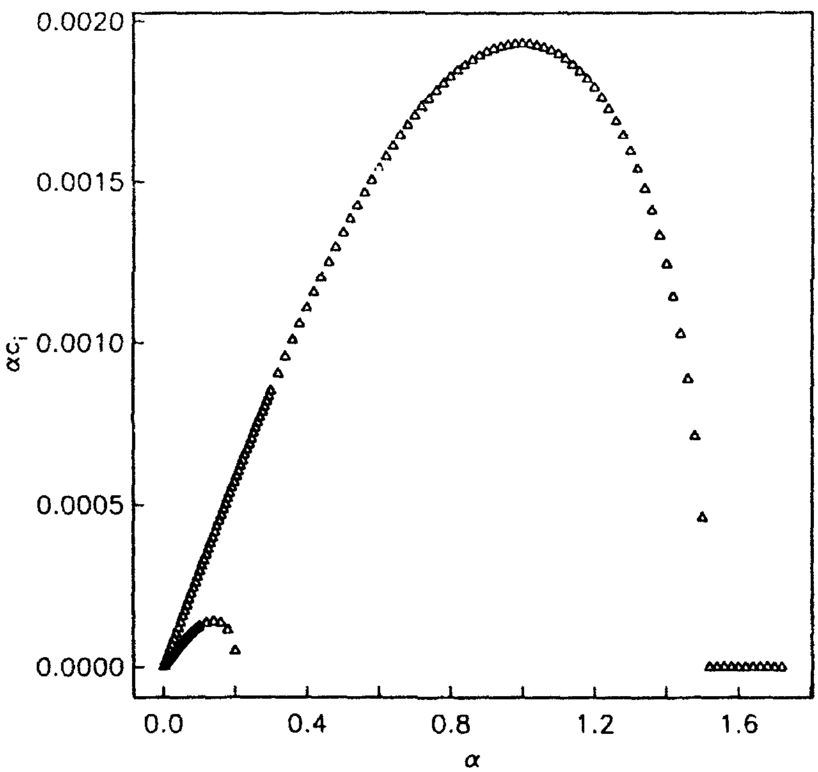

Figure 1. Growth rates αc_i of the primary and secondary modes

4. Results and Discussion

We will limit our discussion to the secondary instability of the nonlinear vortex state obtained by Denier & Hall (1993); further results concerning the effect of crossflow on the secondary modes will be presented in an future paper (see Otto & Denier (1993) for full details). The details of the solution of the governing equations for the nonlinear vortex state can be found in Denier & Hall (1993) and Otto & Bassom (1993); the reader is referred to the aforementioned papers for a discussion of the numerical scheme used to integrate the nonlinear vortex equations.

In figure 1 we present the temporal growth rate of the two major modes which were found. The modes marked 'primary' are conjectured to be projections of the primary vortex onto the reduced inviscid equations. The disturbances marked as 'secondary' are the modes that we are interested in. They have substantially greater growth rates and thus are likely to be more dangerous. The latter modes are characterized by the amount of energy present in the second harmonic.

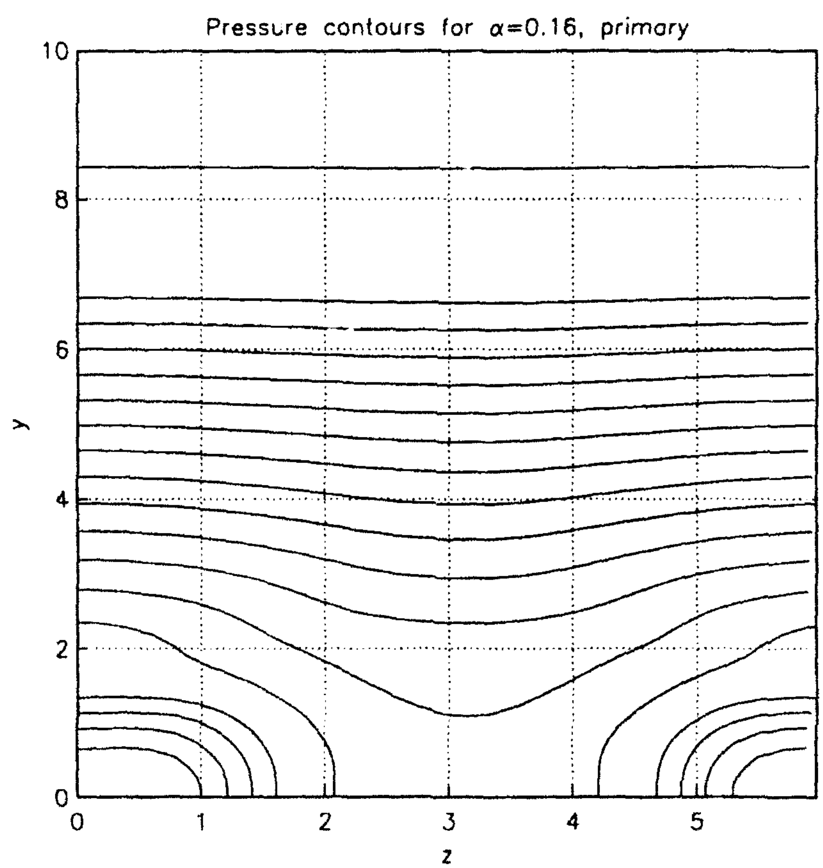

Figure 2a. Iso–pressure contours for the case $\alpha = 0.16$ (primary).

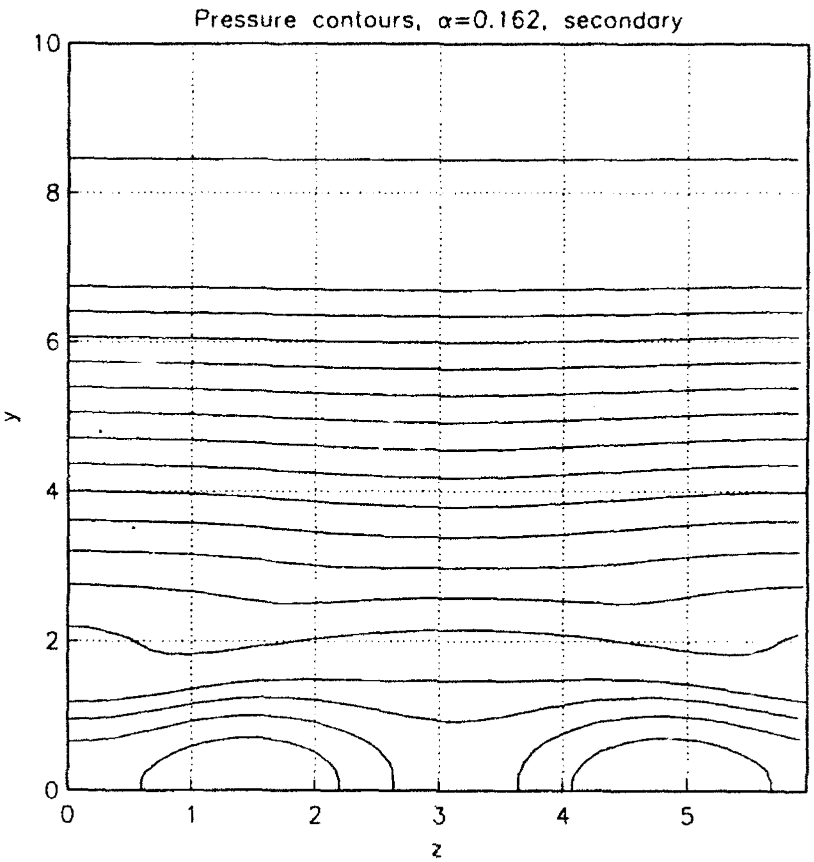

Figure 2b. Iso–pressure contours for the case $\alpha = 0.16$ (secondary).

In figure 2a we show the contours of $|\hat{P}|$ for the case $\alpha = 0.16$ for the primary disturbance, it should be noted that this mode seems to reside near the vortex. In figure 2b we show the contours of pressure for the case $\alpha = 0.16$ for the true secondary disturbance. Notice that this mode has maxima just 'outside' the vortex, (as α increases this phenomena increases in clarity). In Figure 3a,b we present the Fourier decomposition of the two modes, notice these modes are comprised virtually entirely of cosine components, and hence we do not show the sine coefficients. In Figure 3a we note that the majority is confined to the the fundamental and the two-dimensional component (which is not shown), however in figure 3b the mode labeled the 'secondary' disturbance can be seen to have a significant second harmonic component. These results lend credence to the physical significance of the 'primary' and 'secondary' modes pictured in Figure 2.

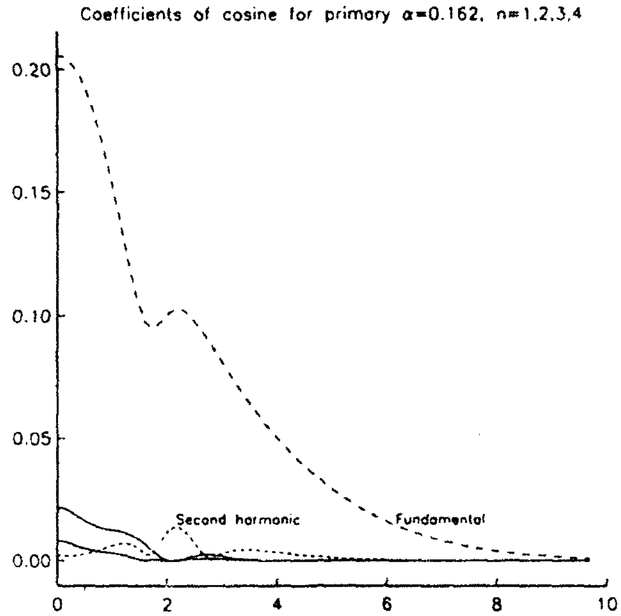

Figure 3a. Fourier decomposition of the pressure (primary).

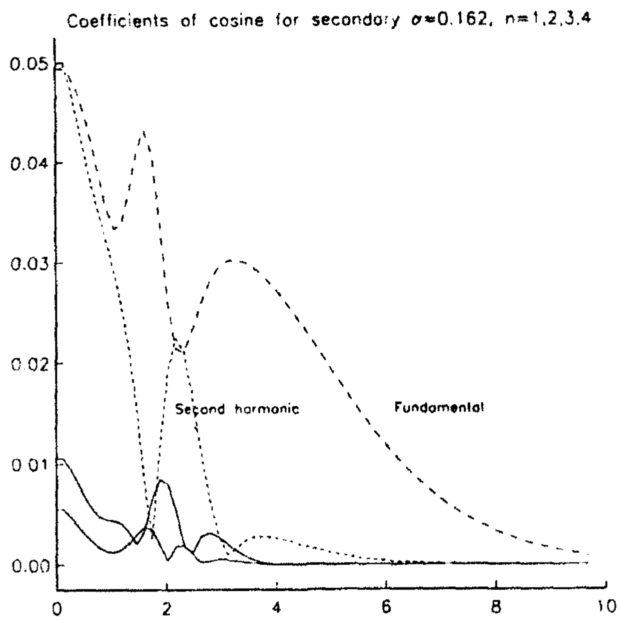

Figure 3b. Fourier decomposition of the pressure (secondary).

By imposing periodicity in the spanwise coordinate we were able to find two distinct modes, here referred to as the primary and secondary mode but were unable to demonstrate the existence of any 'odd' modes which lie $\pi/2$ out-of-phase to the underlying vortex velocity field. However, by removing this requirement and instead imposing boundary conditions appropriate to such odd disturbances we are indeed able to demonstrate the existence of such modes; the full results of this study will be presented in a forthcoming paper (see Otto & Denier (1993)).

5. Conclusions

We have shown that flow situations involving a nonlinear vortex are susceptible to inviscid disturbances with large growth rates. One of the major points that should be noted from this work is that the most unstable secondary modes have significant second harmonic components, which perhaps should be incorporated in the analytic approach to this class of problems. A similar phenomena has been observed in the secondary instability of rotating disk flows by Balachandar, Streett & Malik (1992), the reader is referred to that paper for further insight into the nature of these disturbances.

It is not clear whether flows which are inflectional in nature in the absence of the vortex will be stabilized or destabilized by the introduction of a vortex state; such a question deserves further consideration. Finally, it would be an interesting and significant problem to consider the effect of increasing the amplitude of the inviscid wave to determine whether a vortex/wave interaction could ensue.

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